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OPTIMAL SHAPE DESIGN IN CONTACT PROBLEMS WITH NORMAL COMPLIANCE AND FRICTION

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Abstract—We discuss in this paper the problem of optimal shape design in a certain class of contact problems.

1. INTRODUCTION

We consider the problem of optimal shape design of an elastic body that is in frictional contact with a rigid foundation. The design of such bodies or parts is of industrial interest, for example in the shape of the brakes of a car. The problem is to choose a shape of a part out of a prescribed family of admissible shapes such that, after elastic deformation, the stresses on the contact area are more evenly spread. This should guarantee longer lifetime and better reliability for the part, compared to the cases where high stresses are localized.

We deal with a static problem of frictional contact with normal compliance contact law introduced by Oden and Martins [1] (see also [2]), and appropriately modified Coulomb's law of dry friction. Such problems were considered recently in [1–8] (see also [9] and references therein). Questions of existence, uniqueness, numerical solutions and other mathematical properties were investigated there. Problems of optimal shape design with prescribed friction bound were investigated in [10]. Here, we follow their method. Recently, Klarbring [11] has considered the problem of optimizing the distribution of contact forces without friction in the finite dimensional or discretized cases. His main interest has been in the appropriate choice of the objective function.

We use the standard notation for problems with friction, see e.g., [2,9,10]. Let U_{ad} be a set of functions $\alpha = \alpha(x)$, described below, such that $\Omega(\alpha) \subset \mathbf{R}^2$ is the reference configuration of an elastic body, the shape of which depends on α as a parameter (see Figure 1). We assume that $\partial\Omega(\alpha) = \bar{\Gamma}_o \cup \bar{\Gamma}_t \cup \bar{\Gamma}(\alpha)$ for each $\alpha \in U_{ad}$, where Dirichlet data is prescribed on $\bar{\Gamma}_o$, the surface tractions $\mathbf{t}$ are prescribed on Γ_t, and $\Gamma(\alpha)$ is the potential contact surface. If $\mathbf{u} = \mathbf{u}(\alpha) = (u_1(x,y;\alpha), u_2(x,y;\alpha))$ is the displacement vector, then the static problem can be written as a variational inequality as follows. Let $V_\alpha = \{\mathbf{v} \in \mathbf{H}^1(\Omega(\alpha)) \colon \mathbf{v} = 0 \text{ on } \Gamma_o\}$ be the set of admissible displacements. Let

$$a(\mathbf{u},\mathbf{v}) = \int\limits_{\Omega(\alpha)} a_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l}\, dx \tag{1}$$

be a bilinear coercive and continuous form, $a(\cdot,\cdot)\colon V_\alpha \times V_\alpha \to \mathbf{R}$, representing the elastic energy, where a_{ijkl}, $i,j,k,l = 1,2$, are the elasticity coefficients.

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Next, let $F\colon V_\alpha \to \mathbf{R}$ be the linear functional

$$F(\mathbf{v}) = \int_{\Omega(\alpha)} \mathbf{f} \cdot \mathbf{v}\, dx + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}\, ds, \tag{2}$$

where $\mathbf{f}$ are the body forces acting on $\Omega(\alpha)$, while $\mathbf{t}$ are the surface tractions. F is the potential energy of the forces. The *normal compliance* functional $P : H^{1/2}(\Gamma(\alpha)) \times H^{1/2}(\Gamma(\alpha)) \to \mathbf{R}$ is defined by

$$P(\xi,\eta) = \int_{\Gamma(\alpha)} C_N[(\xi - g)_+]^{m_N}\, \eta\, ds, \tag{3}$$

and represents the energy of surface penetration. Here, g is the distance between the surface $\Gamma(\alpha)$, and the foundation. In the cases that we consider, since the foundation is a straight line (see Figure 1), we have that $g = \alpha$. Finally, the *friction functional* $j : H^{1/2}(\Gamma(\alpha)) \times H^{1/2}(\Gamma(\alpha)) \to \mathbf{R}$ is given by

$$j(\xi,\eta) = \int_{\Gamma(\alpha)} C_T[(\xi - g)_+]^{m_T}\, |\eta|\, ds. \tag{4}$$

The variational formulation of the frictional contact problem is (see [2,9])

$$(P_\alpha)\begin{cases} \text{find } \mathbf{u}(\alpha) \in V_\alpha \text{ such} & \text{that } \forall \mathbf{v} \in V_\alpha \\ a(\mathbf{u},\, \mathbf{u}) + P(u_N, u_N) & + j(u_N, u_T) + F(\mathbf{u}) \\ & \le a(\mathbf{u},\, \mathbf{v}) + P(u_N, v_N) + j(u_N, v_T) + F(\mathbf{v}). \end{cases} \tag{5}$$

Here, u_N, u_T are the normal and tangential components of $\mathbf{u}$ on $\Gamma(\alpha)$, respectively.
We assume that $\mathbf{f} \in L^2(\Omega(\alpha))$, $\mathbf{t} \in L^2(\Gamma_t)$, $a_{ijkl} \in L^\infty(\Omega(\alpha))$ $i,j,k,l = 1,2$ and

$$a_{ijkl}\xi_{ij}\xi_{kl} \ge \lambda|\xi|^2, \qquad \forall\xi \text{ such that } \xi_{ij} = \xi_{ji},$$

$$a_{ijkl} = a_{jikl} = a_{klij}.$$

Then, by a result of [2], for each $\alpha \in U_{ad}$, problem (P_α) has a solution.

2. OPTIMAL SHAPE DESIGN

We turn to the problem of the optimal shape design. Consider the family of shapes $\Omega(\alpha) \in O$, such that $\Gamma(\alpha)$ can be represented as the graph of $y = \alpha(x), a \le x \le b$, see Figure 1. Therefore, $g = \alpha$ in (3) and (4).

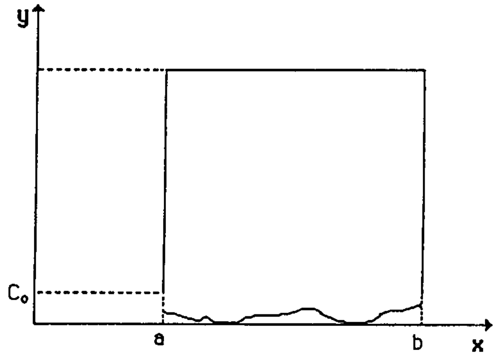

Figure 1. The family $\Omega(\alpha)$ is parametrized by α.

The parametrization is such that $\alpha \in U_{ad}$, where

$$U_{ad} = \{\alpha \in W^{\kappa,p}(a,b); 0 \le \alpha \le C_0, \|\alpha\|_{W^{\kappa,p}(a,b)} \le C_1\} \tag{6}$$

for $\kappa > 1, p \ge 2$ and $C_0 < \gamma$. Then, $O = \{\Omega(\alpha)\}_{\alpha \in U_{ad}}$ and

$$\Omega(\alpha) = \{(x,y) \in \mathbf{R}^2 : a < x < b, \alpha(x) < y < \gamma\},\ \alpha \in U_{ad}.$$

Also, let

$$\hat{\Omega} = \{(x,y) \in \mathbf{R}^2 : a < x < b, 0 < y < \gamma\}, \tag{7}$$

so that $\Omega(\alpha) \subset \hat{\Omega}$ for all $\alpha \in U_{ad}$.

It is known that $W^{\kappa,p}(a,b)$ is compactly imbedded in $W^{1,1}(a,b)$ and in $C^{\lambda}([a,b])$ for any, $\lambda < \kappa - 1/p$ (see e.g., [12, p. 1,48]).

The *optimal shape design* problem is

$$(P_{OSD}) \begin{cases} \text{find } \alpha^* \in U_{ad} \text{ such that} \\ \\ J(\alpha^*) = \min\limits_{\alpha \in U_{ad}} J(\alpha). \end{cases} \tag{8}$$

Here, $J = J(\alpha)$ is a cost functional, which measures the optimality of the shape. We consider one such cost functional below and prove the existence of a minimizer for (P_{OSD}).

3. THE "ENERGY" COST FUNCTIONAL

We consider the cost functional analogous to that in [10] (see also references there), namely

$$\begin{aligned} J(\alpha) \equiv J(\alpha, \mathbf{u}(\alpha)) = \frac{1}{2}\, a(\mathbf{u}(\alpha), \mathbf{u}(\alpha)) + P(u_N(\alpha), u_N(\alpha)) \\ + j(u_N(\alpha), u_T(\alpha)) + F(\mathbf{u}(\alpha)). \end{aligned} \tag{9}$$

To show that problem (P_{ODS}) with this cost functional has a solution, we use an abstract theorem of [10]. We need the following assumptions.

$$[A1] \qquad \text{Let } G = \{(\Omega(\alpha), \mathbf{u}(\alpha))\} = \{(\alpha, \mathbf{u}(\alpha)) : \alpha \in U_{ad}\}$$

be *compact* in the following sense: if $\{\alpha_n\} \subset U_{ad}$ is an arbitrary sequence, then there exists a subsequence $\{(\alpha_{n_k}, \mathbf{u}\,(\alpha_{n_k})\}$ and an element $(\bar{\alpha}, \mathbf{u}(\bar{\alpha})) \in G$ such that

$$\alpha_{n_k} \to \bar{\alpha}$$

and

$$\mathbf{u}(\alpha_{n_k}) \to \mathbf{u}(\bar{\alpha})$$

as $k \to \infty$.

The sense of the convergence has to be made precise in each application.

[A2] J is *lower semicontinuous:* if $\alpha_n, \alpha \in U_{ad}, \alpha_n \to \alpha$ and if $\mathbf{v}_n \in V_{\alpha_n}$ with $\mathbf{v}_n \to \mathbf{v}$, then

$$\liminf_{n \to \infty} J(\alpha_n, \mathbf{v}_n) \ge J(\alpha, \mathbf{v}). \tag{10}$$

THEOREM 1. *Assume that [A1] and [A2] hold. Then, problem (P_{OSD}) has at least one solution.*

PROOF. Theorem 2.1 in [10, p. 29]. ∎

Our main result is the following:

THEOREM 2. *Problem (P_{OSD}) with J given in (9) has a solution.*

PROOF. We have to check that our problem satisfies [A1] and [A2]. This is done in the lemmas below, then the result follows from Theorem 1.

LEMMA 3. *Condition [A1] holds with*

$$\alpha_{n_k} \to \bar{\alpha} \text{ weakly in } W^{\kappa,p}(a,b) \tag{11}$$

and

$$\mathbf{u}(\alpha_{n_k}) \to \mathbf{u}(\bar{\alpha}) \text{ weakly in } \mathbf{H}^1(\hat{\Omega}). \tag{12}$$

PROOF. Equipped with the weak topology of $W^{\kappa,p}(a,b), U_{ad}$ is a closed bounded set and therefore compact in $W^{1,1}_{a,b}$. Hence, there exists a subsequence α_{n_k} converging weakly in $W^{\kappa,p}(a,b)$ to $\bar{\alpha} \in U_{ad}$. Next, we extend the $\mathbf{u}(\alpha_{n_k})$ to $\hat{\Omega}$. Since Korn's constant does not depend on α_{n_k}, (see e.g., [10,13]) thus,

$$\|\mathbf{u}(\alpha_{n_k})\|_{\mathbf{H}^1(\alpha_{n_k})} \leq C.$$

Then, we use the Calderon extension (see e.g., [10]) and obtain $\|\mathbf{u}(\alpha_{n_k})\|_{\mathbf{H}^1(\hat{\Omega})} \leq C$. Therefore, (for a subsequence of $\{\alpha_{n_k}\}$) $\mathbf{u}(\alpha_k) \to \mathbf{w} \in \mathbf{H}^1(\hat{\Omega})$ weakly, for some $\mathbf{w}$. It remains to be shown that $\mathbf{w} = \mathbf{u}(\bar{\alpha})$. We have to limit the various terms in (5), starting with P.

$$\int\limits_{\Gamma(\alpha_k)} C_N\,(u_N(\alpha_k) - \alpha_k)_+^{m_N}\, v_N\, ds = \int_a^b C_N\,(u_N \circ \alpha_k - \alpha_k)_+^{m_N}\, v_N \circ \alpha_k \sqrt{1 + (\alpha_k')^2}\, dx.$$

But $\mathbf{v} \in \mathbf{H}^1(\hat{\Omega})$, and the sequence $\{\alpha_n\}$ converges strongly in $W^{1,1}(a,b)$ and in $C([a,b])$, and we also have

$$\int\limits_a^b \left|u_N \circ \alpha_k - u_N \circ \bar{\alpha}\right| dx \leq \int_a^b \left|\int_{\bar{\alpha}}^{\alpha_k} \frac{\partial v_N}{\partial y}\, dy\right| dx \leq C \left\|\alpha_k - \bar{\alpha}\right\|^{\frac{1}{2}}_{C([a,b])} \left\|v_N\right\|_{\mathbf{H}^1(\hat{\Omega})}$$

and the right hand side converges to zero. We conclude that

$$\int\limits_{\Gamma(\alpha_n)} C_N(u_N(\alpha_k) - \alpha_k)_+^{m_N} u_N ds \;\to\; \int\limits_{\Gamma(\bar{\alpha})} C_N(w_N - \bar{\alpha})_+^{m_N} u_N\, ds.$$

Similarly,

$$\int\limits_{\Gamma(\alpha_n)} C_T(u_N(\alpha_k) - \alpha_k)_+^{m_T}\, |v_T|\, ds \;\to\; \int\limits_{\Gamma(\bar{\alpha})} C_T(w_N - \bar{\alpha})_+^{m_T} |v_T|\, ds.$$

The same arguments show that

$$P(u_N(\alpha_k), u_N(\alpha_k)) \to P(w_N, w_N)$$

and

$$j(u_N(\alpha_k), u_T(\alpha_k)) \to j(w_N, w_T).$$

That $F(\mathbf{u}(\alpha_k)) \to F(\mathbf{u})$ is simple to show. It remains to limit the terms with $a(\cdot,\cdot)$. To this end, for each $m \in \mathbb{N}$ (fixed), let

$$G_m = G_m(\alpha) = \left\{(x,y) \in \mathbb{R}^2 \colon a < x < b,\ \alpha(x) + \frac{1}{m} < y < \gamma\right\}.$$

The uniform convergence $\alpha_k \to \bar{\alpha}$ implies that these exists n_o, such that $\bar{G}_m \subset \Omega(\alpha_k)$ for all $k \geq n_o$. Dividing $\Omega(\alpha_k)$ into three parts $\Omega(\alpha_k) = G_m(\bar{\alpha}) \cup (\Omega(\alpha_k)\backslash\Omega(\bar{\alpha})) \cup ((\Omega(\bar{\alpha})\backslash G_m(\bar{\alpha}) \cap \Omega(\alpha_n))) \equiv G_m \cup I_1 \cup I_2$ leads to the following:

$$a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{\Omega(\alpha_k)} = a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{G_m} +$$
$$a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{I_1} + a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{I_2} \leq a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{G_m} +$$
$$a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{I_1} + a(\mathbf{u}(\alpha_k), \mathbf{v})_{I_2}.$$

Estimating each of the three terms separately, as $k \to \infty$, gives

$$\limsup_{k\to\infty} a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k))_{G_m} \leq a(\mathbf{w},\ \mathbf{v} - \mathbf{w})_{G_m}$$

and

$$\limsup_{k\to\infty} |a(\mathbf{u}(\alpha_k), \mathbf{v})_{I_2}| = 0$$

(see [10, p. 57]). Moreover,

$$\limsup_{k\to\infty} |a(\mathbf{u}(\alpha_k), \mathbf{v})_{I_2}| \leq C\, \|\mathbf{v}\|_{\mathbf{H}^1(I_2)}$$

(see [10, p. 58]). Consequently,

$$\limsup_{k\to\infty} a(\mathbf{u}(\alpha_k), \mathbf{v} - \mathbf{u}(\alpha_k)) \leq a(\mathbf{w},\ \mathbf{v} - \mathbf{w}) + C\, \|\mathbf{v}\|_{\mathbf{H}^1(I_2)}.$$

Combining the estimates above gives

$$a(\mathbf{w},\ \mathbf{v} - \mathbf{w})_{G_m} + P(w_N, v_N - w_N) + j(w_N, u_T) - j(w_N, w_T)$$
$$\geq \int_{G_m} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w})\, dx\, dy + \int_{\Gamma_t} \mathbf{t}(\mathbf{v} - \mathbf{w})\, dx - C\|\mathbf{v}\|_{\mathbf{H}^1(I_2)}.$$

Letting $m \to \infty$, we find that $\mathbf{w}$ is a solution for $P_{\bar{\alpha}}$. Therefore, $\mathbf{w} = \mathbf{u}(\bar{\alpha})$ and the proof is complete.

LEMMA 4. *J given in (9) is lower semicontinuous.*

PROOF. The proof is very similar to the above. Again, we limit each term in P_{α_n}, and use the strong convergence in $\mathbf{W}^{1,1}(a,b)$ and in $C([a,b])$. Thus, [A1] and [A2] hold, and therefore, Theorem 2 has been proven.

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